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ON THE POLYNOMIAL CONVERGENTS OF A POWER SERIES *

BY M. B. PORTER

IF $\sum_0^\infty a_n x^n$ be a power series having a circle of convergence whose radius is r and if $P_n(x) = \sum_0^n a_m x^m$ be its polynomial convergent of degree n. Abel has shown that for any given point x', $|x'| > r$, $|P_n(x')|$ will become definitely infinite if n becomes infinite over a suitable set of integers $n_1, n_2, \cdots$ We shall denote such a set by (n_i). Various questions present themselves in this connection. If $\lim_{n_i=\infty} |P_{n_i}(x')| = \infty$, does this same property hold for all the points of a region in which x' lies and if so what are the shapes of such regions? Moreover since $\sum_0^\infty a_n x^n$ converges uniformly in $|x| < r - \epsilon$ to an analytic function $F(x)$, it follows that $P_n(x)$ will have, when n is large enough, just as many zeroes in $|x| < r - \epsilon$ as $F(x)$ has, i. e., a limited number. Thus the question of the distribution of the remaining zeroes of $P_n(x)$ presents itself. Finally it is possible that $\lim P_n(x)$ exist throughout a region lying outside or containing the circle of convergence inside of it when n becomes infinite over the set (n_j). In the last case, when the regions overlap we can get an analytic extension of our power series merely by inserting parentheses.

While the general solution of the foregoing questions would doubtless be a very difficult matter, the following theorem applicable to all power series throws some light on the character of the results we may expect.

THEOREM. If $\sum_0^\infty a_n x^n$ *have a circle of convergence of radius* $r (0 \leqq r \leqq M)$, *then it will always be possible to find a set of integers* (n_j), *such that* $\lim_{n_j=\infty} P_{n_j}(x)$ *becomes definitely infinite throughout any given region lying outside the circle of convergence, save perhaps at a finite number of points of the region.*

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* Presented at the meeting of the American Mathematical Society at Columbia, Mo., December 1, 1906.

If s denote any point outside the circle of convergence, and we consider the numbers $|a_0|$, $|a_1 s|$, $|a_2 s^2|$, $\cdots$ $|a_n s^n|$, $\cdots$, since some of these numbers increase indefinitely, it will be possible to reach one greater than all that preceed it; if we go on still farther in the set, we shall reach a second greater than all that preceed it, and so on indefinitely. Let us denote the numbers thus obtained by $|a_{n_1} s^{n_1}|$, $|a_{n_2} s^{n_2}|$, $\cdots |a_{n_i} s^{n_i}|$, $\cdots$.

Consider the polynomials

$$[1] \qquad P_{n_i}(x) \equiv a_{n_i} x^{n_i}\left[\frac{a_0}{a_{n_i}} x^{-n_i} + \frac{a_1}{a_{n_i}} x^{-n_i+1} + \cdots + \frac{a_{n_i-1}}{a_{n_i}} x^{-1} + 1\right]$$

$$\equiv a_{n_i} x^{n_i} \bar{P}_{n_i}(x) \qquad (i = 1, 2, 3, \cdots).$$

Throughout the region $R(|x| > |s|)$,

$$|\bar{P}_{n_i}(x)| < \frac{1}{1 - \left|\frac{s}{x}\right|}$$

and is consequently limited in the region $R'(|x| > |s| + \epsilon)$. Thus by a theorem of Arzelà's[*] we can pick out of the set (n_i) at least one sequence (n_j) such that the polynomials $\bar{P}_{n_j}(x)$ converge uniformly to a function $\bar{P}(x)$ analytic in R', as n_j becomes infinite. Evidently this function $\bar{P}(x)$ does not vanish identically in R since $\bar{P}_{n_j}(\infty) = 1$.

Thus by a well known theorem, each polynomial $\bar{P}_{n_j}$ will have only a limited number of zeroes (say m') in R' (when n_j is large enough) and these zeroes will condense on the m' zeroes of $\bar{P}(x)$ which lie in R'. From [1] it is clear that these m' zeroes will lie in the region $|s| < |x| \leqq 2|s|$.[†] Denote by m the number of zeroes of $\sum_0^\infty a_n x^n$ in $|x| < r - \epsilon$; then $P_{n_j}(x)$ will have exactly m zeroes in this region if n_j is large enough. Consider a set of points $[s_i]$ where $|s_{i+1}| < |s_i| < |s|$ and $\lim_{i=\infty} s_i = r$. To each s_{i+1} will correspond a set of polynomials which will be a *sub*set of the set corresponding to s_i and at the same time a subset of $[P_{n_j}]$. Thus we see that

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[*] For a proof of this theorem see a paper by Arzelà, ANNALS OF MATHEMATICS, ser. 2, vol. 5 (1903), p. 53, or a paper by the author, vol. 6, p. 190, of the same journal.

[†] If $\lim_{n=\infty} \frac{a_{n+1}}{a_n}$ exist, it is easy to see from elementary considerations that $P_n(x)$ will have no roots in the region $|x| > r + \epsilon$ for a sufficiently large value of n.

From the set $[P_{n_j}]$ *we can always extract a set which will ultimately have all its zeroes, save* $m + m'$, *in the region* $r - \epsilon < |x| < |s_i|$. *Moreover there must be points* on the circle of convergence which are limiting points of the zeroes of these polynomials.* This does not exclude the possibility of there being other limiting points of the complete set $[P_{n_j}]$ in $r - \epsilon < |x| < |s_i|$ *not* on the circle of convergence.

We can go further and show that some of the polynomials $P_{n_j}(x)$ will take on values less than any number small at pleasure in the neighborhood of any point of the circle of convergence. Consider the set of functions $\phi_{n_j} = \frac{1}{P_{n_j}}$ in the region $|x| > r$. If any region surrounding a point on the rim of the circle of convergence existed throughout which $|\phi_{n_j}(x)| < M$ (fixed) for all values of the sequence n_j, then since for the part of this region in which $|x| > r + \epsilon$

$$\lim_{n_j = \infty} \phi_{n_j}(x) = 0,$$

it would follow that

$$\lim_{n_j = \infty} \phi_{n_j} = 0$$

inside the circle $|x| < r$ (by Arzelà's or Stieltje's theorem†), which is a contradiction.

As has already been remarked, it may happen that a sequence $(\bar{n}_i)$ can be found such that $\lim_{\bar{n}_i = \infty} P_{\bar{n}_i}(x)$ exists throughout regions outside the circle of convergence or throughout regions in which the circle of convergence lies.

To construct such series, write

$$[A] \qquad \sum_0^\infty c_{n_i} z^{n_i} \quad \text{(radius of convergence } r\text{)}.$$

Set in this series

$$z = Kx(1 + x);$$

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* In case $r = 0$ then the point $x = 0$ will be such a limiting point; in the case $\sum_0^\infty x^n$ every point on the rim of the circle of convergence is a limiting point.

† For a statement of this theorem see a paper by Osgood, *Functions defined by Infinite Series*, ANNALS OF MATHEMATICS, ser. 2, vol. 3 (1901), p. 33.

then the circle of convergence $|z| = r$ will go over into the Cassinian $r = |Kx(1+x)|$ inside of which

$$[A'] \qquad \phi(x) = \sum_0^\infty c_{n_i} K^{n_i} x^{n_i} (1+x)^{n_i}$$

will converge absolutely. Any singularity of $[A]$ will go over into two singularities of $[A']$. Let us now suppose that $n_{i+1} > 2n_i$; then if the binomials in $[A']$ are multiplied out, it can be written as a power series in x merely by removing parentheses. The resulting power series will converge in a circle about the origin reaching up to the nearest singularity of $\phi(x)$. Evidently in the case before us this circle must lie wholly inside the Cassinian and hence it can only reach up to the nearest point on the Cassinian, which must consequently be a singularity of $\phi(x)$. Similarly if we make the transformation

$$[B] \qquad z = Ke^{\theta i} x(1+x),$$

the same argument shows that *every* point on the circle of convergence $|z| = r$ goes over into a singular point of $\phi(x)$, so that the circle of convergence of $[A]$ is a natural boundary.*

The series $\phi(x)$ regarded as a power series in x is such that by the insertion of suitable parentheses it can be made to converge throughout the domain of existence of the function it defines. Moreover if the Cassinian consists of two ovals, it will converge in both ovals and define different analytic functions, in the sense that neither can be regarded as the analytic continuation of the other.

It may be noticed that the transformation $[B]$ used in constructing $\phi(x)$ reduces the investigation of singularities on the rim of the circle of convergence to a question concerning double series, so that to show that $re^{\theta i}$ is a singularity of $[A]$ it would be sufficient to prove that $[A']$ arranged as a power series in x converges for any positive real value for which it converged before it was so arranged.

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* This also follows from a theorem of Hadamard's; *La Série de Taylor*, p. 36.